QGS : Quantum groups seminar

Quantum affine algebras and spectral k-matrices

1. The Yang-Baxter equation (YBE)

$R(z) \in \mathcal{A} \otimes \mathcal{A}((z))$ s.t. some algebra

$$R_{12}(z) R_{13}(zw) R_{23}(w) =$$
$$= R_{23}(w) R_{13}(zw) R_{12}(z)$$

$R_{12}(z) = R(z) \otimes 1$ $\quad$ $R_{23}(z) = 1 \otimes R(z)$

$R_{13}(z) = (\mathrm{flip} \otimes 1)(R_{23}(z))$

$R(z) =$ R-matrix with spectral parameter z

2. Quantum integrable systems

$\mathfrak{a} = \mathrm{end}(\mathbb{C}^2)$

$q \in \mathbb{C}^\times$ – quantum parameter

$$R(z) := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{q^{-1}(z-1)}{z-q^{-2}} & \dfrac{1-q^{-2}}{z-q^{-2}} & 0 \\ 0 & \dfrac{z(1-q^{-2})}{z-q^{-2}} & \dfrac{q^{-1}(z-1)}{z-q^{-2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

XXZ ½-spin Heisenberg model / 6-vertex model $\rightsquigarrow$ $U_q \widehat{\mathfrak{sl}}_2$

$\circlearrowright$

$\mathbb{C}^2$

3. Quantum affine algebras

$\mathfrak{g}$ = simple f.d. Lie algebra / $\mathbb{C}$

central

$$\hat{\mathfrak{g}} = \underbrace{\mathfrak{g} \otimes \mathbb{C}[t,t^{-1}]}_{\mathcal{L}\mathfrak{g}} \oplus \mathbb{C}c \oplus \mathbb{C}d$$

affine Lie algebra

(Kac-Moody algebra)

loop algebra

derivation

$\widehat{\mathfrak{sl}}_2$

$e_1, f_1 \rightsquigarrow e\otimes 1, f\otimes 1$

$e_0, f_0 \rightsquigarrow f\otimes t, e\otimes t^{-1}$

$$U_q\hat{\mathfrak{g}} \supseteq (U_q\hat{\mathfrak{g}})' \twoheadrightarrow (U_q\hat{\mathfrak{g}})' \big/_{q^c=1} \simeq U_q\mathcal{L}\mathfrak{g}$$

Drinfeld-Jimbo quantum group

(usual Kac-Moody presentation)

Quantum loop algebra

(Drinfeld loop presentation)

4. Representations

$U_q\hat{\mathfrak{g}}$	$U_q L\mathfrak{g}$
$\rightsquigarrow$	$\rightsquigarrow$
$\mathcal{O}^{int}$	$\mathrm{Rep}^{fd}(U_q L\mathfrak{g})$
• $\dim(V) = \infty$	• $\dim(V) < \infty$
• semisimple	• non-semisimple !
• $\underline{\lambda} \in \mathbb{N}^I$	• $P(u) \in \mathbb{C}[u]^I$
• braided	• non-braided !

<u>Remark</u> $U_q L\mathfrak{g}$ is $\mathbb{Z}$-graded ($\deg(e_i/f_i) = 0$, $\deg(e_0/f_0) = \pm 1$) and endowed with

$$(a \in \mathbb{C}^\times) \quad \tau_a \circlearrowleft U_q L\mathfrak{g}$$

$$\tau_a(x) = a^m x \quad \text{if } \deg(x) = m$$

$$V \in \mathrm{Rep}^{fd}(U_q L\mathfrak{g}) \rightsquigarrow V(a) := \tau_a^*(V)$$

Thm (Drinfeld)

$$\exists\ R(z) \in \left(U_q L\mathfrak{g} \,\hat{\otimes}\, U_q L\mathfrak{g}\right)[\![z]\!] \quad \text{s.t.}$$

- $R_{VW}(z) \in \mathrm{end}(V \otimes W)[\![z]\!]$
- $\mathrm{flip} \circ R_{VW}(z)$ intertwiner
- $\Delta \otimes 1\,(R(z)) = R_{13}(z)\, R_{23}(z)$

  $1 \otimes \Delta\,(R(z)) = R_{13}(z)\, R_{12}(z)$

$\rightsquigarrow$ $R_{VV}(z)$ satisfies (YBE).

idea: $U_q\hat{\mathfrak{g}}$ is quasi triangular Hopf algebra

$$R = q^{\Omega_0} \cdot \sum_{\mu > 0} e_\mu \otimes f_\mu \in U_q \hat{n}_+ \,\hat{\otimes}\, U_q \hat{n}_-$$

$$\Omega_0 = \sum u_i \otimes u^i \in \mathfrak{h}$$

$$+ c \otimes d + d \otimes c$$

$$\rightsquigarrow \quad R_{VW}(z) := \tau_z \otimes 1\,(R) \circlearrowleft V \otimes W$$

5. Reflection equation

From integrable models with boundary conditions (Sklyanin, Cherednik, ...)

$k(z) \in \mathcal{A}((z))$ s.t.

$$k_1(z)\, R_{21}^{\varphi}(zw)\, k_2(w)\, R(w/z)$$

$$= R_{21}^{\varphi\varphi}(w/z)\, k_2(z)\, R^{\varphi}(zw)\, k_1(w)$$

$\rightsquigarrow k(z) =$ k-matrix with spectral parameter z

<u>examples</u>

- many computational examples for $U_q L\mathfrak{g}$ (e.g. Delius - George, Regelski - Vlaar ...)

- universal constant solutions for finite-type $U_q\mathfrak{g}$ (Bao - Wang, Balagovic - Kolb)

6. Quantum symmetric pairs

In both cases, the construction of the k-matrix is deeply related to Letzter-Kolb quantum symmetric pairs

$$B \subseteq U_q\mathfrak{g} \leftsquigarrow \mathfrak{g} \text{ Kac-Moody}$$

coideal $\Delta(B) \subseteq B \otimes U_q\mathfrak{g}$

$$(q=1) \quad B \longrightarrow U(\mathfrak{g}^\theta)$$

$$\theta = \mathrm{Ad}(w_X) \circ \omega \circ \tau$$

finite-type Dynkin subdiagram

diagram automorphism with $\tau^2 = 1$ and $\tau|_X =$ action induced by w_X

longest element in W_X

$\mathcal{B} \subseteq U_q\mathfrak{g}$ controls the intertwining property of k on $V \in \mathcal{O}^{int}$:

$\forall v \in V,\ b \in \mathcal{B}$

$$k_V \cdot b \cdot v = \varphi(b) \cdot k_V \cdot v$$

some diagram automorphism

$$\Longrightarrow \quad k_V : V \xrightarrow{\sim} V^{\varphi} \quad \text{as } \mathcal{B}\text{-modules}$$

In affine type, the intertwining equation involves the parameter inversion, i.e.

$$k_V(z) : V(z) \longrightarrow V(1/z)^{\varphi}$$

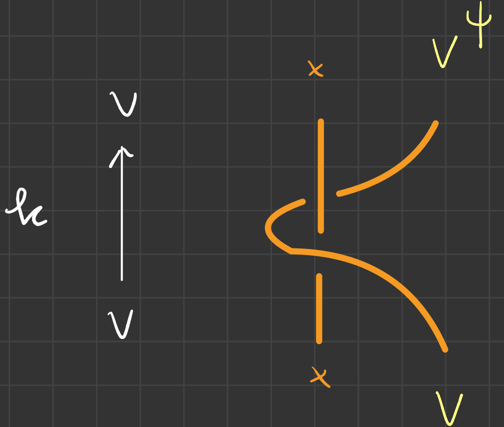
Goal: obtain spectral k-matrices from constant (Kac-Moody) k-matrices.

Big problem: φ cannot perform the parameter inversion

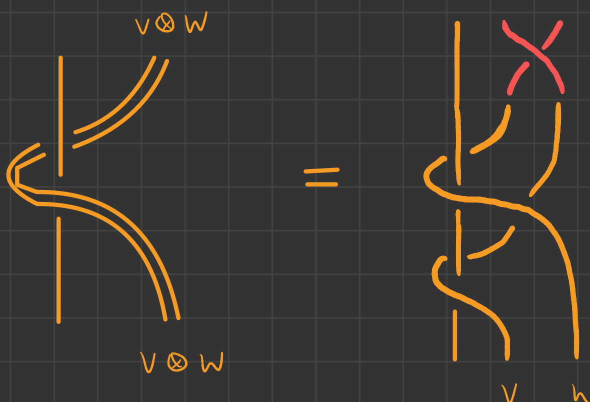
7. A new approach (A. - Vlaar)

- (H, R) = qt Hopf algebra
- $\psi : H \longrightarrow H$ = algebra automorphism
- $V \in Rep(H) =: \mathcal{C}$

We would like an operator $k : V \longrightarrow V^{\psi}$



and we want to get the RE from a suitable coupling identity



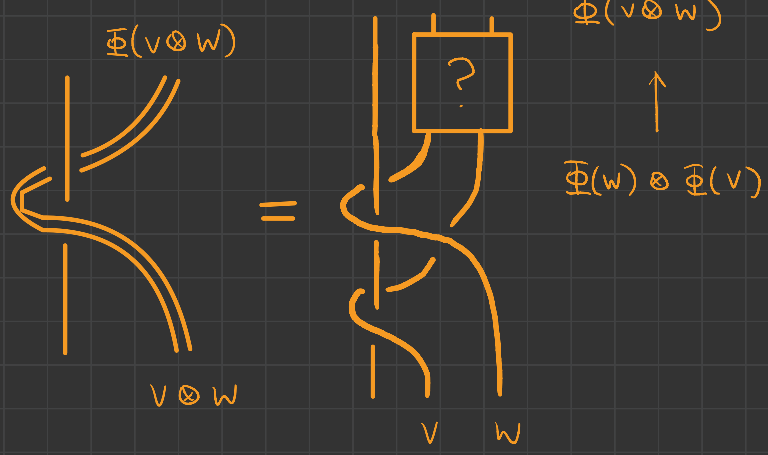
⟵ not very canonical

$\psi = id, \varphi$

Assume instead we have a functor

$\Phi : \mathcal{C} \longrightarrow \mathcal{C}$ codifying the twisting so that

$k : V \to \Phi(V)$ (say $\Phi = \varphi^*$) and



Our choice becomes almost canonical once we

assume Φ is a braided functor

$$(\Phi, J) : \mathcal{C} \longrightarrow \mathcal{C}^{op}$$

opposite braided monoidal category

$$\boxed{J : \Phi(V \otimes W) \xrightarrow{\sim} \Phi(W) \otimes \Phi(V)}$$

precisely what we need !

Algebraically, our new setting is given by the following data:

- (H, R) = quasitriangular Hopf algebra
- $\psi : H \to H$ = algebra automorphism
- $J \in H \otimes H$ = Drinfeld twist

satisfying

$$(H^{cop})^{\psi} = H_J \qquad R_{21}^{\psi\psi} = R_J$$

where

$(H^{\psi}, R^{\psi\psi})$ is the ψ-twisted qtHA

$$\Delta^{\psi}(x) := \psi \otimes \psi \circ \Delta \circ \psi^{-1}(x)$$

$$R^{\psi\psi} := \psi \otimes \psi (R)$$

(H_J, R_J) is the J-twisted qtHA

$$\Delta_J(x) := J \cdot \Delta(x) \cdot J^{-1}$$

$$R_J := J_{21} \cdot R \cdot J^{-1}$$

We assume (H, R) is fixed and we call (ψ, J) a twist pair. Note that this is equivalent to have a braided equivalence

$$(\psi^*, J) : \mathrm{Rep}(H, R) \longrightarrow \mathrm{Rep}(H, R)^{op} = \mathrm{Rep}(H^{cop}, R_{21})$$

Finally we get the k-matrix.

<u>Def</u> A cylindrical bialgebra is a tuple (H, R, ψ, J, k) as before with $k \in H$ s.t.

$$\Delta(k) = J^{-1} \cdot 1 \otimes k \cdot R^{\psi} \cdot k \otimes 1$$

new coupling identity

Cylindrical bialgebras provide the right framework to study a more general reflection equation:

(1) $R^{\psi\psi}_{21} \cdot 1 \otimes k \cdot R^{\psi} \cdot k \otimes 1 =$

$$= J_{21} \cdot R \cdot \underbrace{J^{-1} \cdot 1 \otimes k \cdot R^{\psi} \cdot k \otimes 1}_{\Delta(k)} =$$

$$= k \otimes 1 \cdot R^{\psi}_{21} \cdot 1 \otimes k \cdot R$$

(2) k, R induce an external action of the cylindrical braid group on $V^{\otimes N}$

(3) Set $\vartheta(x) := k^{-1} \cdot \psi(x) \cdot k$ and let $B_k \subseteq H^{\vartheta}$ be the maximal coideal subalgebra. Then, $\mathrm{Rep}(B_k)$ is a braided module category over $\mathrm{Rep}(H)$.

8. New k-matrices for $U_q\mathfrak{g}$

Let $U_q\mathfrak{g}$ be a quantum Kac-Moody algebra and $B \subseteq U_q\mathfrak{g}$ a QSP corresponding to $\vartheta = \mathrm{Ad}(w_X) \circ \omega \circ \tau$.

The construction of a k-matrix for $U_q\mathfrak{g}$ supported on B requires the use of quantum Weyl group operators:

T_i = q-analogues of triple exponentials

[L, KR, LS] action of B_W on $U_q\mathfrak{g}$ and $\mathcal{O}^{int}$ representations

[Kamnitzer-Tingley] $\exists\ t_X := (\text{Cartan})\, T_{w_X}$

s.t.

$$R_X := t_X^{-1} \otimes t_X^{-1} \cdot \Delta(t_X)$$

t_X is a half-balance for $U_q\mathfrak{g}_X$

Modified quantum Weyl group operators

Given Satake diagrams (X, τ) and (Y, η) define suitable corrections

$$T_{X,\tau} := (\text{Cartan}) \cdot T_{w_X}$$

$$R_{X,\tau} := (\text{Cartan}) \cdot R_X$$

and we set

$$\vartheta_q(X,\tau) := \mathrm{Ad}(T_{X,\tau}) \circ \omega \circ \tau \circlearrowright U_q\mathfrak{g}$$

$$\psi_{Y,\eta} := \mathrm{Ad}(T_{Y,\eta}) \circ \omega \circ \tau$$

such that $(\psi_{Y,\eta}, R_{Y,\eta})$ is a twist pair.

Rem in finite type, we choose $Y = I$ and $\psi = \mathrm{op}_I \circ \tau$ (cf. Balagović - Kolb).

Finally we can state our

Thm (A. - Vlaar)

There exists a canonical series $\mathcal{X}_{X,\tau} = 1 + \sum_\mu \mathcal{X}_\mu$ with $\mathcal{X}_\mu \in (U_q \mathfrak{n}_+)_\mu$ s.t.

$$k_{Y,\eta} := \left(T_{Y,\eta}^{-1} \cdot T_{X,\tau} \right) \cdot (\text{Cartan}) \cdot \mathcal{X}_{X,\tau}$$

is a k-matrix for the twist pair $(\psi_{Y,\eta}, R_{Y,\eta})$ with support $\mathcal{B}_{X,\tau}$.

Affine type $Y = I \setminus \{0, \tau(0)\}$ and $\eta(0) = \tau(0)$ $(0 \notin X)$ and $\eta|_Y = \mathrm{op}_Y$. For any $V \in \mathrm{Rep}^{fd}\, U_q L\mathfrak{g}$, we get a formal series

$$k_{Y,\eta}(z) \in \mathrm{end}(V)[[z]]$$

satisfying the spectral reflection equation in $\psi_{Y,\eta}$.